Section 3-1 Numeration Systems

A numeral is a symbol that is used to represent a number. A collection of numerals along with properties that represent numbers systematically is a numeration system.

The Hindu-Arabic system we use is also called a base ten system or decimal system, since it is based on powers of 10.

An important part of any numeration system is place value. Place value assigns a value to a digit based on the digit's location in the numeral.

13,562

ten thousands 10^4, thousands 10^3, hundreds 10^2, tens 10^1, ones $10^0 = 1$

In expanded form this is:

$$1 \cdot 10^4 + 3 \cdot 10^3 + 5 \cdot 10^2 + 6 \cdot 10^1 + 2 \cdot 1$$

Roman Numeration System

(covered in 4th grade, sometimes 3rd)

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

The Roman numeration system is an addition system. So, XII = 10+1+1=12

However, when one symbol of lesser value is immediately to the left of a symbol of greater value, subtraction is implied. So, IX = 10-1 = 9

ex 1 Translate to a Hindu-Arabic numeral.

a) $CC\underbrace{XL}_{\text{sub}}\underbrace{IV}_{\text{sub}} = 100+100+(50-10)+(5-1) = 244$

b) $DCLXX\underbrace{IX}_{\text{sub}} = 500+100+50+10+10+(10-1) = 679$

c) $M\underbrace{CD}_{\text{sub}}LXIII = 1000+(500-100)+50+10+3$

$= 1463$

ex 2 Write as a roman numeral.

a) 99 = IC

b) 685 = DCLXXXV

c) 474 = CDLXXIV

Note: A bar is used to denote multiplication by 1,000. Two bars represents multiplication by 1,000,000 (that's 1,000 twice!)

ex: $\overline{\text{IX}} = 9 \cdot 1{,}000 = 9{,}000$

$\overline{\text{L}}\,\text{IV} = 50 \cdot 1{,}000 + 4 = 50{,}004$

Other bases are used in computer science: binary (base two) and hexadecimal (base 16).

Let's look at other bases, starting with base five.

Base Five

Digits in base five: 0, 1, 2, 3, 4

Place values in base five:

ones	5^0
fives	5^1
twenty-fives	5^2
125s	5^3
⋮	

Counting in base five:

$1_{five}, 2_{five}, 3_{five}, 4_{five}, 10_{five}, 11_{five}, \ldots$

10_{five}

fives ones

In base ten, 10.

ones

tens

Let's keep counting:

2 grps of 5

$10_{five}, 11_{five}, 12_{five}, 13_{five}, 14_{five}, 20_{five}, \ldots$

I regroup with every five.

ex 3 Fill in the missing terms in the counting sequence.

a) 32_{five}, 33_{five}, 34_{five}, $\underline{40_{five}}$, $\underline{41_{five}}$

b) $\underline{43_{five}}$, $\underline{44_{five}}$, 100_{five}, 101_{five}, 102_{five}, ...

25 5 1

c) $\underline{14_{five}}$, 20_{five}, 21_{five}, 22_{five}, $\underline{23_{five}}$, $\underline{24_{five}}$

ex 4 Convert to base ten by using the expanded form.

a) 431_{five} $\quad 4\cdot5^2+3\cdot5^1+1\cdot1 = \boxed{116}$

b) 100_{five} $\quad 1\cdot5^2+0\cdot5^1+0\cdot1 = \boxed{25}$

c) 24_{five} $\quad 2\cdot5^1+4\cdot1 = \boxed{14}$

ex 5 Convert the following base ten number to base 5.
(Note: no base written means base ten.)

a) 23 $5^0, 5^1, \cancel{5^2}, \cancel{5^3}$

largest power smaller than 23

$5^1 = 5$ | 23 → 4
-20
$5^0 = 1$ | 3 → 3
-3
0

$\boxed{43_{five}}$

* if you write 43_{five} in expaned form and multiply it all out, you should get 23.

b) 140 $5^0, 5^1, 5^2, 5^3$

$5^3 = 125$ | 140 → 1 125s $\boxed{1030_{five}}$
-125
$5^2 = 25$ | 15 → 0 25s
- 0
$5^1 = 5$ | 15 → 3 fives
- 15
$5^0 = 1$ | 0 → 0 ones

Section 3-1 Part 2

Let's look at other bases and extend our knowledge about base five.

Base Three

Digits: 0, 1, 2

Place values: $1, 3^1, 3^2, 3^3, 3^4, \ldots$

1 grp of 3

Counting: $1_{three}, 2_{three}, 10_{three}, 11_{three}, 12_{three},$

1 grp of 9

$20_{three}, 21_{three}, 22_{three}, 100_{three}, 101_{three}, \ldots$

ex 6 Convert to base ten. (Use expanded form)

a) 220_{three}

$$2 \cdot 3^2 + 2 \cdot 3^1 + 0 \cdot 1 = \boxed{24}$$

b) 1001_{three}

$$1 \cdot 3^3 + 0 \cdot 3^2 + 0 \cdot 3^1 + 1 = \boxed{28}$$

ex 7 Convert to base three. (Use division)

a) 17 $\quad 3^0, 3^1, 3^2, \cancel{3^3}$

$3^2 = 9$	17 -9	1
$3^1 = 3$	8 -6	2
$3^0 = 1$	2 -2	2
	0	

$\boxed{122_{three}}$

b) 100 $\quad 3^0, 3^1, 3^2, 3^3, 3^4, \cancel{3^5}$

$3^4 = 81$	100 -81	1
$3^3 = 27$	19 -0	0
$3^2 = 9$	19 -18	2
$3^1 = 3$	1 -0	0
$3^0 = 1$	1 -1	1
	0	

$\boxed{10201_{three}}$

Other Bases

ex 8 Fill in the blanks of the following counting sequences.

a) 12_{four}, 13_{four}, $\underline{20_{four}}$, $\underline{21_{four}}$, ...

base 4
0, 1, 2, 3

b) $\underline{7_{eight}}$, 10_{eight}, 11_{eight}, 12_{eight}, ...

base 8
0, 1, 2, 3, 4, 5, 6, 7

c) 4_{seven}, 5_{seven}, 6_{seven}, $\underline{10_{seven}}$, $\underline{11_{seven}}$, ...

base 7
0, 1, 2, 3, 4, 5, 6

d) $\underline{88_{nine}}$ 100_{nine}, 101_{nine}, 102_{nine}, ...

base 9
0, 1, 2, 3, 4, 5, 6, 7, 8

We can even talk about bases higher than 10.

Digits in base 2: 0, 1
" 3: 0, 1, 2
" 4: 0, 1, 2, 3
" 5: 0, 1, 2, 3, 4
" 6: 0, 1, 2, 3, 4, 5
" 7: 0, 1, 2, 3, 4, 5, 6
" 8: 0, 1, 2, 3, 4, 5, 6, 7
" 9: 0, 1, 2, 3, 4, 5, 6, 7, 8
" 10: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
" 11: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T
" 12: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E

ten eleven

* the T = 10 and E = 11 in base twelve

ex9 Convert to base ten. (expanded form)

a) 406_{seven}

$4 \cdot 7^2 + 0 \cdot 7^1 + 6 \cdot 1 = \boxed{199}$

b) 101001_{two}

$1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 1$

$= \boxed{41}$

c) $93T6_{eleven}$

$9 \cdot 11^3 + 3 \cdot 11^2 + 10 \cdot 11 + 6 \cdot 1 = \boxed{12,458}$

d) $TE2_{twelve}$

$10 \cdot 12^2 + 11 \cdot 12^1 + 2 \cdot 1 = \boxed{1,574}$

ex10 Convert to the indicated base. (Division)

a) 26 to base two $\quad 2^0, 2^1, 2^2, 2^3, \underset{16}{2^4}, \underset{32}{\cancel{2^5}}$

$2^4 = 16$	26	1
	-16	
$2^3 = 8$	10	1
	-8	
$2^2 = 4$	2	0
	-0	
$2^1 = 2$	2	1
	-2	
$2^0 = 1$	0	0
	-0	

$\boxed{11010_{two}}$

* remember, you can check these by using expanded form to convert back to base ten.

b) 1742 to base seven

$7^0, 7^1, 7^2, 7^3 = 343, \cancel{7^4} = 2401$

$7^3 = 343$ 1742   5

−1715

$7^2 = 49$ 27   0

− 0

$7^1 = 7$ 27   3

− 21

$7^0 = 1$ 6   6

−6

0

$\boxed{5036_{seven}}$

c) 1277 to base twelve

$12^0, 12^1, 12^2 = 144, \cancel{12^3 = 1728}$

$12^2 = 144$ 1277   8

−1152

$12^1 = 12$ 125   10 = T

−120

$12^0 = 1$ 5   5

− 5

0

$\boxed{8T5_{twelve}}$

end 3-1